Abstract signatures for substitution systems

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Abstract (1/2)

A new semantical/categorical notion of signature is introduced that is intended to capture all kinds of syntax with variable binding in the style of inductive families, which could also be called locally nameless or typeful de Bruijn representation. In particular, the new notion encompasses the notion of signature introduced by the author and Uustalu that has a strength-like datum and is used to study heterogeneous substitution systems. The latter are based on the presentation of monads with monad multiplication that is notoriously difficult to generalize to the setting of relative monads.

Abstract (2/2)

Abstract signatures allow definitions relative to a given functor, so that variable names of terms can also be taken from a restricted set, such as the natural numbers. The module concept of Hirschowitz and Maggesi (however, rather the relativization by Ahrens) is adapted through “proto-modules” that come with less data than modules, which is crucial in the construction process.

This is work in progress.

Outline

The papers alluded to in the abstract are:

- the notion of heterogeneous substitution systems based on a notion of signature with strength (not called as such in the paper) [MU04]
- the modules over a monad [HM10]
- the modules over a relative monad [Ahr16]

This work should be carefully compared with a workshop paper by Hirschowitz and Maggesi [HM12].

This work is about generic mechanisms to introduce syntax for languages involving variable binding, such as the lambda calculus. A notion of substitution is seen as essential ingredient of syntatic structures. This notion should be provided "generically". In systems with (a very general form of) variable binding, it may not be obvious what the right notion of substitution would be. (For pure untyped lambda calculus, there is no such problem, but in richer languages such as with explicit flattening, the situation is not as obvious.)

In [MU04], the central question was how to specify concretely in what sense substitution for a given system is a “homomorphism”, taking into account all of the binding. In other words, the main question was the specification of the behaviour of substitution on the term constructors of the object calculus. This is embodied in the notion of a heterogeneous substitution system (HSS), and it is proved that any HSS yields a monad. The bind operation of the monad represents parallel substitution in the object calculus, and the monad laws ensure general “hygiene” conditions, while not addressing in any form the behaviour on the term constructors of the object calculus. However, that part is taken care of by the HSS.

In short, in [MU04], a (generic) construction of a monad is described from a notion of signature that is expressed in terms of category theory and has a semantic flavour. This has been formalized [AM16] in UniMath. A more concrete construction provided a monad on univalent sets from a simple syntactic description in form of a binding signature [AMM16].

The notion of signature of [MU04] is very general, and it is unlikely that one would like to consider monads coming from syntax that are not covered by that framework. However, the approach is formulated for endofunctors only—in particular, the monad as end result is based on an endofunctor T. While this is part of the definition of a monad, there is also the notion of relative monad [ACU15] that can cope with functors that are not endofunctors.

However, the constructions in [MU04] rely on the existence of monad multiplication $\mu : T \cdot T \to T$, which use self-composition for the functor T already in its signature. Obviously, T can only be composed with itself when T is an endofunctor. Therefore, it is essential to replace monad multiplication by the—for monads equivalent—notion of bind.

We are heading for a generalization of the results of [MU04] to relative monads. As mentioned before, relative monads will need different techniques. We are not heading for capturing more examples that yield monads. Hence, we would be satisfied if the extension was “conservative”. However, we are not striving for only a conservative extension and try to see which generality is possible by developing the new notions.

A generalization of monads are modules over monads [HM10]. Monads are in particular modules over themselves. If a monad over the endofunctor $Z : \mathcal{C} \to \mathcal{C}$ comes with an operation bind consisting of, for all $C_1, C_2 \in \mathcal{C}$, operations

$$\mathrm{bind}_{C_1,C_2} : \mathcal{C}(C_1, ZC_2) \to \mathcal{C}(ZC_1, ZC_2) \ ,$$

a module over that module has an endofunctor $X : \mathcal{C} \to \mathcal{C}$ and an operation mbind with, for all $C_1, C_2 \in \mathcal{C}$,

$$\mathrm{mbind}_{C_1,C_2} : \mathcal{C}(C_1, ZC_2) \to \mathcal{C}(XC_1, XC_2) \ .$$

(Here, we deliberately omit the other ingredients and the laws.)

If we are to construct the syntax of an object language with variable binding as endofunctor T on $\mathcal{C}$, we might want to find it as fixed point for X of the equation $A + HXA \cong XA$ for an endofunctor H on $[\mathcal{C}, \mathcal{C}]$ and all $A \in \mathcal{C}$ (assuming binary coproducts in $\mathcal{C}$). This can be written as one single isomorphism equation $\mathrm{Id} + HX \cong X$.

In this line of work, we are interest in the construction of monads, not in working with them, so the question arises if a partially constructed binding operation can be successively extended in the construction process. If we are to construct a bind operation successively, we would be facing the three occurrences of Z in its type. We might assume we had it already for $Z := X$, and we would like to extend it to $Z := \mathrm{Id} + HX$, maybe by first giving it for $Z := HX$. However, this visibly changes the type of the argument to bind as well, and no such lifting process can be given. To the contrary, mbind operations allow just that: if we have it for X, we can try to lift it to $\mathrm{Id} + HX$ in place of X, possibly by passing through HX in place of X, and this with always the same type of the arguments to the different instances of mbind.

A fixed-point construction should hence give us an mbind operation with $X := T$, the solution to the fixed-point equation. But what is Z then?

The problem for us with the notion of module over a monad is that we are in the construction process of a module T that should in the end become our monad, but the notion of module is based on a monad for Z, that we want to instantiate with T in the end.

In the present work, we identify substructures of monads and modules over monads that allow to specify requirements and to carry out the construction process that, in the end, yields a monad. Since our original motivation was to get rid of the endofunctor restriction, we are heading for a relative monad and have to devise substructures of relative monads and of modules over relative monads. As before for the relative monads, this is mainly a question of type-checking the definitions.

Let categories $\mathcal{C}$, $\mathcal{D}$ be given. We fix a functor $J : \mathcal{C} \to \mathcal{D}$ and consider notions relative to that functor. However, the case of $\mathcal{D} = \mathcal{C}$ and $J = \mathrm{Id}$ is an important source of examples.

Definition (Relative proto-monad)

A relative proto-monad consists of a functor $Z : \mathcal{C} \to \mathcal{D}$ and a natural transformation $e : J \to Z$.

In other words, a relative proto-monad is a “pointed functor” relative to J. In particular, there is no bind operation.

Definition (Relative proto-module)

Given a relative proto-monad (Z, e) and a category $\mathcal{E}$, a relative proto-module over (Z, e) with codomain $\mathcal{E}$ consists of a functor $X : \mathcal{C} \to \mathcal{E}$ and an operation $\mathrm{mbind} = (\mathrm{mbind}_{C_1,C_2})_{C_1,C_2 \in \mathcal{C}}$ with

$$\mathrm{mbind}_{C_1,C_2} : \mathcal{D}(JC_1, ZC_2) \to \mathcal{E}(XC_1, XC_2)$$

for all $C_1, C_2 \in \mathcal{C}$, such that

$$\mathrm{mbind}_{C,C}(e_C) = \mathrm{id}_{XC} \qquad (\eta\text{-mbind})$$

holds for all $C \in \mathcal{C}$, and such that mbind is "natural" in the following (natural) sense: for all $C_1, C_1', C_2, C_2' \in \mathcal{C}$, $h_1 : C_1' \to C_1$, $h_2 : C_2 \to C_2'$ (the latter two in $\mathcal{C}$) and $f : JC_1 \to ZC_2$ (in $\mathcal{D}$), one has

$$Xh_2 \circ \mathrm{mbind}_{C_1,C_2}(f) \circ Xh_1 = \mathrm{mbind}_{C_1',C_2'}(Zh_2 \circ f \circ Jh_1) \quad (\text{n-mbind})$$

Currently, we have no use of the generality that $\mathcal{E}$ may be different from $\mathcal{D}$.

Lemma

Let (X, mbind) *be a relative proto-module over the relative proto-monad* (Z, e)*. Then, for all* $C_1, C_2 \in \mathcal{C}$ *and* $f : C_1 \to C_2$*, one has*

$$Xf = \mathrm{mbind}_{C_1,C_2}(e_{C_2} \circ Jf) \qquad \text{(f-mbind)}$$

as $\mathcal{E}$*-morphisms from* XC_1 *to* XC_2*.*

Notice that the conclusion of this lemma also holds for relative modules, introduced in [Ahr16], but for different reasons (in particular, X is not even assumed to be a functor there).

Proof Combine n-mbind (with $h_2 := \mathrm{id}_{C_2}$, $h_1 := f$ and $f := e_{C_2}$) and η-mbind. □

In the following definition, we look at the non-relativized case ($\mathcal{C} = \mathcal{D}$ and $J = \mathsf{Id}$).

Definition (Proto-module action)

Let a proto-monad (Z, e) and a proto-module (X, mbind) over (Z, e) with codomain $\mathcal{E}$ be given. The proto-module action of (X, mbind) is the natural transformation $\mu^{\mathsf{m}} : X \cdot Z \to X$ that is defined pointwise for $C \in \mathcal{C}$ by

$$\mu^{\mathsf{m}}_C := \mathsf{mbind}_{ZC,C}(\mathsf{id}_{ZC}) \ .$$

Proof Naturality follows immediately from n-mbind. □

The name "action" stems from the definition of modules in [HM10]. It cannot be formulated in a simple way in the relative setting.

Definition (Relative premonad)

Let (Z, e) be a relative proto-monad. A relative premonad over (Z, e) is a triple (Z, e, bind), where $\mathrm{bind} = (\mathrm{bind}_{C_1,C_2})_{C_1,C_2 \in \mathcal{C}}$ with

$$\mathrm{bind}_{C_1,C_2} : \mathcal{D}(JC_1, ZC_2) \to \mathcal{D}(ZC_1, ZC_2)$$

for all $C_1, C_2 \in \mathcal{C}$, such that

$$\mathrm{bind}_{C,C}(e_C) = \mathrm{id}_{ZC} \qquad (\eta\text{-bind})$$

holds for all $C \in \mathcal{C}$ and

$$\mathrm{bind}_{C_1,C_2}(f) \circ e_{C_1} = f \qquad (\beta\text{-bind})$$

holds for all $C_1, C_2 \in \mathcal{C}$ and $f : JC_1 \to ZC_2$, and such that bind is "natural".

Naturality is meant above in the following (again natural) sense: for all $C_1, C_1', C_2, C_2' \in \mathcal{C}$, $h_1 : C_1' \to C_1$, $h_2 : C_2 \to C_2'$ (the latter two morphisms in $\mathcal{C}$) and $f : JC_1 \to ZC_2$ (in $\mathcal{D}$), one has

$$Zh_2 \circ \mathrm{bind}_{C_1,C_2}(f) \circ Zh_1 = \mathrm{bind}_{C_1',C_2'}(Zh_2 \circ f \circ Jh_1) \qquad \text{(n-bind)}$$

as $\mathcal{D}$-morphisms from ZC_1' to ZC_2'.

In particular, we do not assume a law about composition of bind operations.

Lemma

Let (Z, e, bind) *be a relative premonad. Then, for all* $C_1, C_2 \in \mathcal{C}$ *and* $f : C_1 \to C_2$*, one has*

$$Zf = \mathrm{bind}_{C_1,C_2}(e_{C_2} \circ Jf) \qquad \text{(f-bind)}$$

as $\mathcal{D}$*-morphisms from* ZC_1 *to* ZC_2*.*

Notice that the conclusion of this lemma also holds for relative monads, introduced in [ACU15] (journal version), but for different reasons (again, Z is not even assumed to be a functor there).

Proof Combine n-bind (with $h_2 := \mathrm{id}_{C_2}$, $h_1 := f$ and $f := e_{C_2}$) and η-bind. □

In the following definition, we look at the non-relativized case ($\mathcal{C} = \mathcal{D}$ and $J = \mathsf{Id}$).

Definition (Premonad multiplication)

For a premonad (Z, e, bind), premonad multiplication is the morphism μ from $(Z \cdot Z, e \cdot e)$ to (Z, e) (as pointed endomorphisms) that is defined pointwise for $C \in \mathcal{C}$ by

$$\mu_C := \mathsf{bind}_{ZC,C}(\mathsf{id}_{ZC}) \enspace .$$

Proof Naturality follows immediately from n-bind, and for the morphism property, we calculate

$$\mu_C \circ (e \cdot e)_C = \mu_C \circ ((Z \cdot e) \circ e)_C$$

$$= (\mathsf{bind}_{ZC,C}(\mathsf{id}_{ZC}) \circ Ze_C) \circ e_C = \mathsf{bind}_{C,C}(e_C) \circ e_C$$

by n-bind, and the latter term is equal to e_C by applying either η-bind or β-bind. □

Definition (Relative premodule)

Let (Z, e, bind) be a relative premonad. A relative proto-module (X, mbind) over the relative proto-monad (Z, e) with codomain $\mathcal{E}$ is a relative premodule over (Z, e, bind) with codomain $\mathcal{E}$ if, forall $C_1, C_2, C_3 \in \mathcal{C}$, $f : JC_1 \to ZC_2$ and $g : JC_2 \to ZC_3$ (the latter two morphisms in $\mathcal{D}$), one has

$$\text{mbind}_{C_2,C_3}(g) \circ \text{mbind}_{C_1,C_2}(f) = \text{mbind}_{C_1,C_3}\big(\text{bind}_{C_2,C_3}(g) \circ f\big) \qquad (\pi\text{-mbind})$$

as $\mathcal{E}$-morphisms from XC_1 to XC_3.

Lemma

A relative premonad (Z, e, bind) *is a relative monad on* J *(in the sense of [ACU15]) iff forall* $C_1, C_2, C_3 \in \mathcal{C}$, $f : JC_1 \to ZC_2$ *and* $g : JC_2 \to ZC_3$ *(the latter two morphisms in* $\mathcal{D}$*), one has*

$$\mathrm{bind}_{C_2,C_3}(g) \circ \mathrm{bind}_{C_1,C_2}(f) = \mathrm{bind}_{C_1,C_3}\big(\mathrm{bind}_{C_2,C_3}(g) \circ f\big) \quad (\pi\text{-bind})$$

as $\mathcal{D}$*-morphisms from* ZC_1 *to* ZC_3. □

Remark

As remarked after Definition 2.1 in [ACU15], the operation Z *of a relative monad is functorial, and the operations* e *and* bind *of a relative monad are natural (the latter of which is expressed by* n-bind*). Hence, relative monads on* J *are even characterized as relative premonads with the extra condition of the preceding lemma.*

Lemma

If (Z, e, bind) *is a relative monad, then a relative premodule over it is a relative module over it (in the sense of [Ahr16]).* □

A remark similar to the preceding one applies here.

Let again categories $\mathcal{C}$, $\mathcal{D}$ be given, and fix a functor $J : \mathcal{C} \to \mathcal{D}$.

Definition (Morphism of relative proto-monads)

Given relative proto-monads (Z, e) and (Z', e'), a morphism from (Z, e) to (Z', e') is a natural transformation $m : Z \to Z'$ such that $m \circ e = e'$ as natural transformations from J to Z'.

Not surprisingly, this is the notion of morphisms of pointed functors (relative to J). Of course, identity and composition are inherited from $[\mathcal{C}, \mathcal{D}]$, and we get a category.

Definition (Base change)

Given a morphism m of relative proto-monads from (Z, e) to (Z', e') and a relative proto-module (X, mbind) over (Z', e') with codomain $\mathcal{E}$, define the relative proto-module $m^*(X, \mathrm{mbind})$ over (Z, e) with codomain $\mathcal{E}$ as (X, mbind^*), setting for all $C_1, C_2 \in \mathcal{C}$, $f : JC_1 \to ZC_2$,

$$\mathrm{mbind}^*_{C_1,C_2}(f) := \mathrm{mbind}_{C_1,C_2}(m_{C_2} \circ f)$$

Proof The properties η-mbind and n-mbind are readily verified.

□

The analogous definition for relative modules appears under the name “pullback module” in [Ahr16]. The name “base change” was used in [HM10] concerning the action-based definition of modules.

Definition (Morphism of relative proto-modules)

Given a relative proto-monad (Z, e), a category $\mathcal{E}$, and relative proto-modules (X, mbind) and (X', mbind') over (Z, e) with codomain $\mathcal{E}$, a morphism from (X, mbind) to (X', mbind') is a natural transformation $n : X \to X'$ such that for all $C_1, C_2 \in \mathcal{C}$, $f : JC_1 \to ZC_2$, one has

$$n_{C_2} \circ \mathrm{mbind}_{C_1,C_2}(f) = \mathrm{mbind}'_{C_1,C_2}(f) \circ n_{C_1}$$

as $\mathcal{E}$-morphisms from XC_1 to $X'C_2$.

Identity and composition are inherited from $[\mathcal{C}, \mathcal{E}]$, and we get a category. Note that the equational condition is that of morphisms of relative modules [Ahr16], but we also require n to be natural.

Definition (Big category of relative proto-modules)

Given a category $\mathcal{E}$, define a category by taking as objects pairs of a relative proto-monad (Z, e) and a relative proto-module (X, mbind) over (Z, e) with codomain $\mathcal{E}$, and as morphisms from $((Z, e), (X, \mathrm{mbind}))$ to $((Z', e'), (X', \mathrm{mbind}'))$ pairs (m, n) of a morphism $m : (Z, e) \to (Z', e')$ of relative proto-monads and a morphism $n : (X, \mathrm{mbind}) \to m^*(X', \mathrm{mbind}')$ of relative proto-modules over (Z, e) with codomain $\mathcal{E}$. Identity and composition are inherited from the constituent categories. Let the category be called $\mathrm{BProtoModules}_{\mathcal{E}}$.

The definition in [HM12] of the big module category is thus slightly adapted to our situation of relative proto-modules. Let P_1 be the forgetful functor from $\mathrm{BProtoModules}_{\mathcal{E}}$ to the category of proto-monads.

Definition (Morphism of relative premonads)

Given relative premonads (Z, e, bind) and (Z', e', bind'), a morphism from (Z, e, bind) to (Z', e', bind') is a morphism m of relative proto-monads from (Z, e) to (Z', e') such that for all $C_1, C_2 \in \mathcal{C}$, $f : JC_1 \to ZC_2$, one has

$$m_{C_2} \circ \mathrm{bind}_{C_1,C_2}(f) = \mathrm{bind}'_{C_1,C_2}(m_{C_2} \circ f) \circ m_{C_1}$$

as $\mathcal{D}$-morphisms from ZC_1 to $Z'C_2$.

Identity and composition are inherited from the morphisms of relative proto-monads, and we get a category. Note that the equational conditions are those of relative monads [ACU15], but we also require m to be natural.
Morphisms of relative premodules are just inherited from the latter being relative proto-modules.

Let categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be given. We fix a functor $J : \mathcal{C} \to \mathcal{D}$ and an endofunctor H on $[\mathcal{C}, \mathcal{E}]$. We will omit mentioning in the following definitions that they are seen relative to J.

Definition

A proto-lifting for H is an operation $(\cdot)^H$ that assigns to any relative proto-monad (Z, e) and any relative proto-module (X, mbind) over (Z, e) with codomain $\mathcal{E}$ a relative proto-module (HX, mbind^H) over (Z, e).

Notice that the notion mbind^H does not show the dependency on (Z, e).

Definition

A proto-lifting $(\cdot)^H$ is a prelifting for H iff for any relative premonad (Z, e, bind) and any relative premodule (X, mbind) over (Z, e, bind) with codomain $\mathcal{E}$, the assigned relative proto-module (HX, mbind^H) over (Z, e) is a relative premodule over (Z, e, bind).

The following functor has been studied in the past to capture explicit flattening in syntax.

Example (Self-composition)

Consider the endofunctor H_{flat} on $[\mathcal{C}, \mathcal{C}]$ that is given on objects by $H_{\text{flat}} X := X \cdot X$. Given the relative proto-monad (Z, e) and the relative proto-module (X, mbind) over (Z, e) with codomain $\mathcal{C}$, we define mbind^H pointwise as follows:

$$\text{mbind}^H_{C_1,C_2}(f) := \text{mbind}_{XC_1,XC_2}\big(e_{XC_2} \circ (\text{mbind}_{C_1,C_2}(f))\big)$$

for $f : C_1 \to ZC_2$, as $\mathcal{C}$-morphism from $X(XC_1)$ to $X(XC_2)$.
One readily verifies that η-mbind and n-mbind carry over from mbind to mbind^H. Thus, the assignment is a proto-lifting for H_{flat}. One can even show that it is a prelifting for H_{flat}.

The following functor is used to represent lambda abstraction in a typeful de Bruijn representation.

Example (Abstraction)

Assume that $\mathcal{C}$ has binary coproducts, witten as $+$, and fix an object C_0 of $\mathcal{C}$ (often, this will be a terminal object). Consider the functor option : $[\mathcal{C}, \mathcal{C}]$, given on objects by option $C := C_0 + C$. Consider the endofunctor H_{abs} on $[\mathcal{C}, \mathcal{C}]$ that is given on objects by $H_{\mathrm{abs}} X := X \cdot \mathrm{option}$. Given the relative proto-monad (Z, e) and the relative proto-module (X, mbind) over (Z, e) with codomain $\mathcal{C}$, we define mbind^H pointwise as follows:

$$\mathrm{mbind}^H_{C_1, C_2}(f) := \mathrm{mbind}_{C_0+C_1, C_0+C_2}[e_{C_0+C_2} \circ \mathrm{None}, Z\,\mathrm{Some} \circ f]$$

for $f : C_1 \to ZC_2$, as $\mathcal{C}$-morphism from $X(C_0 + C_1)$ to $X(C_0 + C_2)$.

Example continued

Here, we used the injections None : $C_0 \to C_0 + C_2$ and Some : $C_2 \to C_0 + C_2$ (that are components of natural transformations) and the operator $[\cdot, \cdot]$ for the mediating morphism out of the coproduct.
One readily verifies that η-mbind and n-mbind carry over from mbind to mbind^H. Thus, the assignment is a proto-lifting for H_{abs}.
One can even show that it is a prelifting for H_{abs}.

Example (Signatures with strength)

This (class of) example(s) is endogeneous in nature, with $\mathcal{C} = \mathcal{D} = \mathcal{E}$ and $J = \mathrm{Id}$ and H an endofunctor on $[\mathcal{C}, \mathcal{C}]$. We assume that (H, θ) is a signature in the sense of [MU04], i. e., that θ is a natural transformation $\theta : (H-) \cdot U\sim \to H(- \cdot U\sim)$ between bifunctors $[\mathcal{C}, \mathcal{C}] \times \mathrm{Ptd}(\mathcal{C}) \to [\mathcal{C}, \mathcal{C}]$ such that θ is "linear" in the second component. Here, $\mathrm{Ptd}(\mathcal{C})$ denotes the category of pointed endofunctors of $\mathcal{C}$, and U is the forgetful functor from $\mathrm{Ptd}(\mathcal{C})$ to $[\mathcal{C}, \mathcal{C}]$. "Linearity" means

$$\theta_{X,\mathrm{id}} = \mathrm{id}_{HX} \qquad \text{(id-}\theta\text{)}$$

and

$$\theta_{X,(Z' \cdot Z, e' \cdot e)} = \theta_{X \cdot Z',(Z,e)} \circ (\theta_{X,(Z',e')} \cdot Z) \qquad (\circ\text{-}\theta)$$

for all $X : [\mathcal{C}, \mathcal{C}]$ and $(Z, e), (Z', e') \in \mathrm{Ptd}(\mathcal{C})$, where we write id for the identity functor with its trivial point in $\mathrm{Ptd}(\mathcal{C})$. The second equation is between natural transformations from $HX \cdot Z' \cdot Z$ to $H(X \cdot Z' \cdot Z)$.

Example continued

Given the relative proto-monad (Z, e) and the relative proto-module (X, mbind) over (Z, e) with codomain $\mathcal{C}$, we define mbind^H pointwise as follows:

$$\mathrm{mbind}^H_{C_1,C_2}(f) := (H\mu^{\mathrm{m}} \circ \theta_{X,(Z,e)})_{C_2} \circ HXf$$

for $f : C_1 \to ZC_2$, as $\mathcal{C}$-morphism from HXC_1 to HXC_2, using the proto-module action $\mu^{\mathrm{m}} : X \cdot Z \to X$, see Definition 4.

Notice that the two earlier examples are instances of signatures with strength [MU04], and the given operations mbind^H there are also instances of the present definition.

With more effort, one can show that the assignment is a proto-lifting for H. One can even show that it is a prelifting for H: π-mbind can be established without π-bind, which is usually an essential element for reasoning with monads. It seems that one could even take out either η-bind or β-bind from Definition 5 and still obtain the result.

The motivation for this work was to capture all the signatures with strength, but to formulate the conditions in a way that also makes sense in the relativized setting.

Until now, the operation was considered “pointwise”.

Definition

A proto-lifting for H is natural if the assignment of proto-modules extends to an endofunctor on $\mathrm{BProtoModules}_{\mathcal{E}}$ (that commutes with P_1, i. e., preserves the underlying proto-monad).

A similar notion for modules over monads is called “module transformer” in [HM12].

The construction process for the substitution monad requires naturality, not only on the level of proto-liftings but also on the level of pre-liftings.

Conclusion

- Work in progress
- Further refine the notions of monads and modules for the question of constructing them
- Handle notions relative to a given functor
- Surprise: while lifting of proto-modules is essential for the definition, one can also lift the premodules without requiring the full monad laws

Thank you.

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